

# Displacement exponent for loop-erased random walk on the Sierpiński gasket

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## ABSTRACT

We prove that loop-erased random walks on finite pre-Sierpiński gaskets can be extended to the infinite pre-Sierpiński gasket by virtue of the ‘erasing-larger-loops-first’ method, and obtain the asymptotic behavior of the walk as the number of steps increases, in particular, the displacement exponent and a law of the iterated logarithm.

*Key words:* loop-erased random walk ; displacement exponent ; law of the iterated logarithm ; Sierpinski gasket ; fractal

*MSC2010 Subject Classifications:* 60F99, 60G17, 28A80, 37F25, 37F35

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## 1 Introduction

Loop-erased random walk (LERW) is a process obtained by erasing loops from a simple random walk in chronological order (as soon as each loop is made). LERW was originally considered on  $\mathbb{Z}^d$  and the existence of the scaling limit has been proved for all  $d$ . The asymptotic behavior of the walk has been studied in terms of the growth exponent (expected to be the reciprocal of the displacement exponent). For the growth exponents for LERW on  $\mathbb{Z}^d$ , see, for example, [11], [12], [13], [10] and [15].

In this paper, we consider LERW on the Sierpiński gasket and prove the following Theorems 1–3.

**Theorem 1** *Loop-erased random walks on the finite Sierpiński gaskets can be extended to a loop-erased random walk on the infinite Sierpiński gasket.*

Let  $\lambda = (20 + \sqrt{205})/15$  and  $\nu = \log 2 / \log \lambda$ .

**Theorem 2** For any  $s > 0$ , there exist positive constants  $C_1(s)$  and  $C_2(s)$  such that

$$C_1(s)n^{s\nu} \leq E[|X(n)|^s] \leq C_2(s)n^{s\nu},$$

where  $X(n)$  denotes the location of the LERW starting at the origin after  $n$  steps and  $|\cdot|$  the Euclidean distance.

$\nu$  is called the displacement exponent.

**Theorem 3** There are positive constants  $C_3$  and  $C_4$  such that

$$C_3 \leq \overline{\lim}_{n \rightarrow \infty} \frac{|X(n)|}{\psi(n)} \leq C_4, \text{ a.s.,}$$

where  $\psi(n) = n^\nu(\log \log n)^{1-\nu}$ .

Our main tool for the proof is the ‘erasing-larger-loops-first’ (ELLF) method, which was introduced to study the scaling limit (the limit as the edge length tends to 0). The scaling limit for LERW on the Sierpinski gasket was obtained by two groups independently, using different methods. For the ‘standard’ LERW on general graphs, the uniform spanning tree proves to be a powerful tool ([14]). By ‘standard’, we mean the loops are erased chronologically from a simple random walk as first introduced by G. Lawler ([11]). On the other hand, [4] constructed a LERW on the Sierpiński gasket by ELLF, that is, by erasing loops in descending order of size of loops and proved that the resulting LERW has the same distribution as that of the ‘standard’ LERW. Furthermore, in [5], it is proved that ELLF does work not only for simple random walks, but also for other kinds of random walks on some fractals, in particular, for self-repelling walks on the Sierpiński gasket introduced in [2]. An important reason for this flexibility is that the ELLF method is based on self-similarity of the Sierpiński gasket.

Another advantage of the ELLF method is facilitate the extension of LERW to the infinite Sierpiński gasket by providing us with a natural definition of two series of probability measures on sets of loopless paths. The extension is not trivial, for the simple random walk on the infinite Sierpiński gasket is recurrent. The exact value of the displacement exponent has been known by a scaling argument ([1]). As for the proof of the existence, the authors erroneously wrote in [4] that Theorem 2 has been proved in [14], however, [14] deals with the scaling limit, not LERW on the infinite Sierpiński gasket, and proves the short-time behavior of the limit process  $\overline{X}(t)$ :

**Theorem 4** (Theorem 7.10 in [14]) For any  $p > 0$ , there exist constants  $C_5(p)$ ,  $C_6(p) > 0$  such that for all  $t \in [0, 1]$ ,

$$C_5(p)t^{p\nu} \leq E[|\overline{X}(t)|^p] \leq C_6(p)t^{p\nu},$$

where  $|\overline{X}(t)|$  denotes the Euclidean distance from the starting point at time  $t$  and  $\nu = \log 2 / \log \lambda$ ,  $\lambda = (20 + \sqrt{205})/15$ .

It is expected that the same exponent also rules the long-time behavior of the walk, but the method of proof is different, for one has to look into how the scaled number of steps converges, not only the limit distribution. Thus, the author corrects her error and proves Theorem 2 in this paper.

The first mathematical result on the displacement exponent for a non-Markov random walk on the Sierpiński gasket was obtained in [7], dealing with the ‘standard’ self-avoiding walk, which is defined by the uniform measure on self-avoiding paths of a given length. They showed the existence of the exponent in the form of

$$\lim_{n \rightarrow \infty} \frac{\log E_n[|X'(n)|^s]}{\log n} = s\nu_{SAW}, \quad s > 0 \quad (1.1)$$

where  $|X'(n)|$  denotes the end-to-end distance of an  $n$ -step self-avoiding path, and  $\nu_{SAW} = \log 2 / \log(\frac{7-\sqrt{5}}{2})$ . Since the exponent  $\nu_{SAW}$  is different from  $\nu$  in Theorem 2, the LERW is in a

different universality class from the self-avoiding walk. Note that self-avoiding walk cannot be extended to infinite length, for the consistency condition is not satisfied because of culs-de-sac, thus the expectation is taken over the uniform measure on the  $n$ -step self-avoiding paths. Note also that we have a sharper result in (1.1), which comes from the refinement in the analysis.

The structure of the paper is as follows. In Section 2, we define our notation and in Section 3, we describe the ELLF method of loop-erasing. Section 4 deals with the asymptotics of the exit times from a series of triangles, which is used in Section 6. In Section 5 we extend the walk to the infinite Sierpiński gasket and finally, in Section 6 we prove Theorems 2 and 3.

## 2 Random walk on the pre-Sierpiński gaskets

### 2.1 The pre-Sierpiński gaskets

Let us recall the definition of the pre-Sierpiński gasket: denote  $O = (0, 0)$ ,  $a_0 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ ,  $b_0 = (1, 0)$ ,  $a_N = 2^N a_0$  and  $b_N = 2^N b_0$  for  $N \in \mathbb{N}$ . Let  $F'_0$  be the graph that consists of the three vertices and three edges of  $\triangle Oa_0b_0$  and define a recursive sequence of graphs  $\{F'_N\}_{N=0}^\infty$  by

$$F'_{N+1} = F'_N \cup (F'_N + a_N) \cup (F'_N + b_N), \quad N \in \mathbb{Z}_+ = \{0, 1, 2, \dots\},$$

where  $A + a = \{x + a : x \in A\}$  and  $kA = \{kx : x \in A\}$ .  $F'_0$ ,  $F'_1$  and  $F'_2$  are shown in Fig. 1.

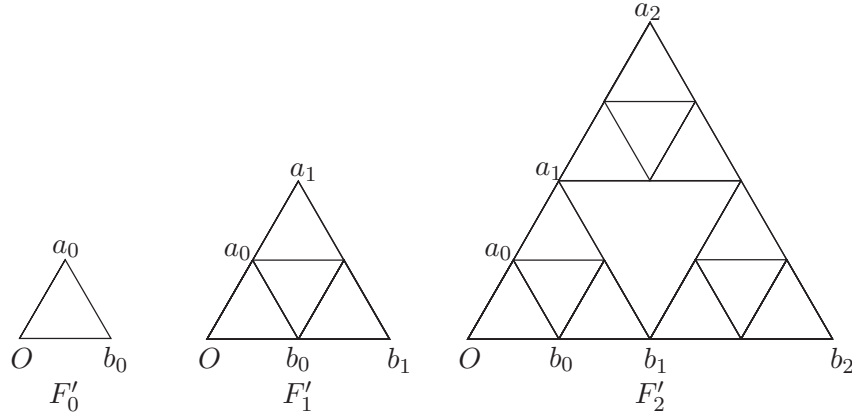


Fig 1:  $F'_0$ ,  $F'_1$  and  $F'_2$ .

Finally, we let  $F_N'^R$  be the reflection of  $F'_N$  with respect to the  $y$ -axis, and denote  $F_0 = \bigcup_{N=1}^\infty (F'_N \cup F_N'^R)$ ; the graph  $F_0$  is called the (infinite) **pre-Sierpiński gasket**.  $F_0$  is shown in Fig. 2.

Furthermore, by letting  $G_0$  and  $E_0$  denote the set of vertices and the set of edges of  $F_0$ , respectively, we see that, for each  $N \in \mathbb{Z}_+$ ,  $F_N = 2^N F_0$  can be regarded as a coarse graph with vertices  $G_N = \{2^N x : x \in G_0\}$  and edges  $E_N = \{2^N(x, y) : (x, y) \in E_0\}$ . We call an upward (closed and filled) triangle which is a translation of  $\triangle Oa_M b_M$  and whose vertices are in  $G_M$  a  **$2^M$ -triangle**.

### 2.2 Paths on the pre-Sierpiński gaskets

Let us denote the set of finite paths on  $F_0$  starting at  $O$  by

$$W = \{ w = (w(0), w(1), \dots, w(n)) : w(0) = O, (w(i-1), w(i)) \in G_0, 1 \leq i \leq n, n \in \mathbb{N} \}.$$

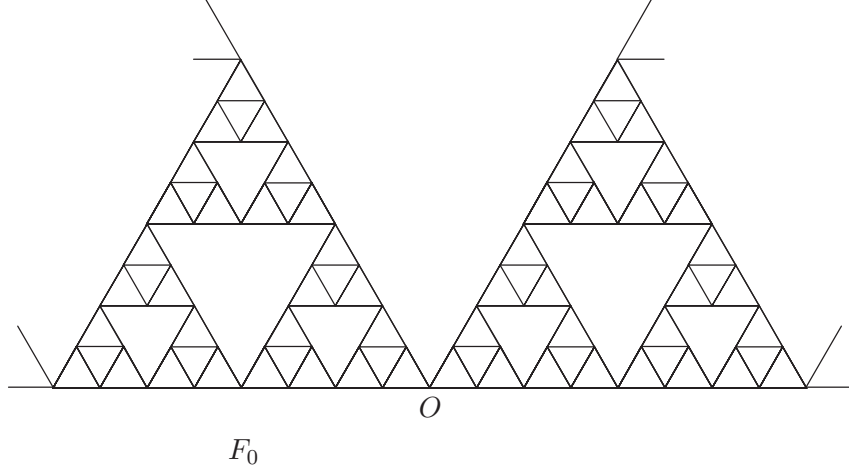


Fig 2: The pre-Sierpiński gasket  $F_0$ .

This gives the natural definition for the length  $\ell$  of a path  $w = (w(0), w(1), \dots, w(n)) \in W$ ; namely,  $\ell(w) = n$ .

For a path  $w \in W$  and  $A \subset G_0$ , we define the hitting time of  $A$  by

$$T_A(w) = \inf\{j \geq 0 : w(j) \in A\},$$

where we set  $\inf \emptyset = \infty$ . By taking  $w \in W$  and  $M \in \mathbb{Z}_+$ , we shall define a recursive sequence  $\{T_i^M(w)\}_{i=0}^m$  of **hitting times of  $G_M$**  as follows: Let  $T_0^M(w) = 0$ , and for  $i \geq 1$ , let

$$T_i^M(w) = \inf\{j > T_{i-1}^M(w) : w(j) \in G_M \setminus \{w(T_{i-1}^M(w))\}\};$$

here we take  $m$  to be the smallest integer such that  $T_{m+1}^M(w) = \infty$ . Then  $T_i^M(w)$  can be interpreted as being the time (steps) taken for the path  $w$  to hit vertices in  $G_M$  for the  $(i+1)$ -st time, under the condition that if  $w$  hits the same vertex in  $G_M$  more than once in a row, we count it only once.

Now, we consider two sequences of subsets of  $W$  as follows: for each  $N \in \mathbb{Z}_+$ , let the set of paths from  $O$  to  $a_N$ , which do not hit any other vertices in  $G_N$  on the way, be

$$W_N = \{w = (w(0), w(1), \dots, w(n)) \in W : w(T_1^N(w)) = a_N, n = T_1^N(w)\},$$

and let the set of paths from  $O$  to  $a_N$  that hit  $b_N$  ‘once’ on the way (subject to the counting rule explained above) be

$$V_N = \{w = (w(0), w(1), \dots, w(n)) \in W : w(T_1^N(w)) = b_N, w(T_2^N(w)) = a_N, n = T_2^N(w)\}.$$

Then, for a path  $w \in W$  and each  $M \in \mathbb{N}$ , we define the **coarse-graining map  $Q_M$**  by

$$(Q_M w)(i) = w(T_i^M(w)), \quad \text{for } i = 0, 1, 2, \dots, m,$$

where  $m$  is the smallest integer such that  $T_{m+1}^M(w) = \infty$  as above. Thus,

$$Q_M w = (w(T_0^M(w)), w(T_1^M(w)), \dots, w(T_m^M(w)))$$

is a path on a coarser graph  $F_M$ . For  $w \in W_N \cup V_N$  and  $M \leq N$ , the end point of the coarse-grained path is  $w(T_m^M(w)) = a_N$ , and if we write  $(2^{-M} Q_M w)(i) = 2^{-M} w(T_i^M(w))$ , then  $2^{-M} Q_M w$  is a path in  $W_{N-M} \cup V_{N-M}$  and  $\ell(2^{-M} Q_M w) = m$ . In the following, we often write  $w(T_i^M)$  instead of  $w(T_i^M(w))$ .

Define a family of probability measures  $P_N$  on  $W_N$ ,  $N = 1, 2, \dots$  by assigning each  $w \in W_N$ ,

$$P_N[w] = \left(\frac{1}{4}\right)^{\ell(w)-1}.$$

$(W_N, P_N)$  defines a family of fixed-end random walks  $Z_N$  on  $F_N$  such that

$$Z_N(w)(i) = w(i), \quad i = 0, \dots, \ell(w), \quad w \in W_N. \quad (2.1)$$

This is a simple random walk on  $F_0$  starting at  $O$  and stopped at the first hitting time of  $a_N$  conditioned that the walk does not hit any vertices in  $G_N \setminus \{O\}$  on the way. The factor  $(1/4)^{-1}$  comes from this conditioning.

Define another family of probability measures  $P'_N$  on  $V_N$ ,  $N = 1, 2, \dots$  by assigning each  $w \in V_N$ ,

$$P'_N[w] = \left(\frac{1}{4}\right)^{\ell(w)-2}.$$

$(V_N, P'_N)$  defines a family of fixed-end random walks  $Z'_N$  on  $F_0$  such that

$$Z'_N(w)(i) = w(i), \quad i = 0, \dots, \ell(w), \quad w \in V_N. \quad (2.2)$$

This is a simple random walk on  $F_0$  starting at  $O$  and stopped at the first hitting time of  $a_N$  conditioned that the walk hits  $b_N$  ‘once’ on the way.

Note that a coarse grained simple random walk is again a simple random walk on a coarse graph, that is, for  $M < N$ ,  $P_N \circ Q_M^{-1} = P_{N-M}$  and  $P'_N \circ Q_M^{-1} = P'_{N-M}$ .

### 3 Loop erasure by the erasing-larger-loops-first rule

For  $(w(0), w(1), \dots, w(n)) \in W_N \cup V_N$ , if there are  $c \in G_0$ ,  $i$  and  $j$ ,  $0 \leq i < j \leq n$  such that  $w(i) = w(j) = c$  and  $w(k) \neq c$  for any  $i < k < j$ , we call the path segment  $[w(i), w(i+1), \dots, w(j)]$  a **loop formed at c** and define its **diameter** by  $d = \max_{i \leq k_1 < k_2 \leq j} |w(k_1) - w(k_2)|$ , where  $|\cdot|$  denotes the Euclidean distance. Note that a loop can be a part of another larger loop formed at some other vertex. By definition, the paths in  $W_N \cup V_N$  do not have any loops with diameter greater than  $2^{N-1}$ . For each  $N \in \mathbb{Z}_+$ , let  $\Gamma_N$  be the set of loopless paths from  $O$  to  $a_N$ :

$$\Gamma_N = \{ (w(0), w(1), \dots, w(n)) \in W_N \cup V_N : w(i) \neq w(j), \quad 0 \leq i < j \leq n, \quad n \in \mathbb{N} \}.$$

Note that any loopless path in  $\Gamma_N$  is confined in  $\triangle Oa_Nb_N$ .

We shall now describe the loop-erasing procedure in a more organized manner than [4]. We start by erasing loops from paths in  $W_1 \cup V_1$ .

#### Loop erasure for $W_1 \cup V_1$

- (i) Erase all the loops formed at  $O$ ;
- (ii) Progress one step forward along the path, and erase all the loops at the new position;
- (iii) Iterate this process, taking another step forward along the path and erasing the loops there, until reaching  $a_1$ .

Denote the resulting path  $Lw$ , where  $L : W_1 \cup V_1 \rightarrow \Gamma_1$  is the loop-erasing operator. Fig. 3 shows all the possible loopless paths from  $O$  to  $a_1$  on  $F_1$ . Here only the parts in  $\triangle Oa_1b_1$  are shown, for any path cannot go into the other triangles without making a loop. Note that  $w \in W_1$  implies  $Lw \in W_1 \cap \Gamma_1$ , but that  $w \in V_1$  can result in  $Lw \in W_1 \cap \Gamma_1$ , with  $b_1$  being erased together

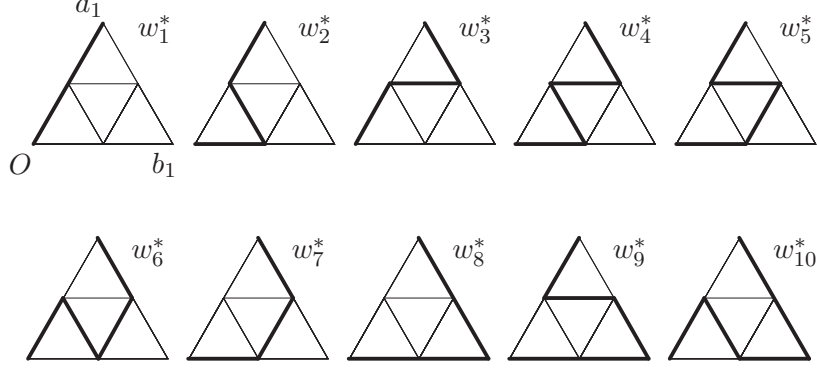


Fig 3: Loopless paths from  $O$  to  $a_1$  on  $F_1$ .

with a loop. So far, our loop-erasing procedure is the same as the chronological method defined for paths on  $\mathbb{Z}^d$  in [11].

For a general  $N$ , we erase loops from the largest-scale loops down, repeatedly applying the loop-erasing procedure for  $W_1 \cup V_1$ . To describe the procedure, we introduce a ‘step-based’ decomposition of a path based on the self-similarity and the symmetries of the pre-Sierpiński gaskets. Assume  $w \in W_N \cup V_N$  and  $0 \leq M < N$ . Note that the pair of adjacent  $2^M$ -triangles including  $(Q_M w)(i-1)$ ,  $(Q_M w)(i)$  and  $(Q_M w)(i+1)$  is similar to  $F_0 \cap (\Delta O a_M b_M \cup \Delta O a_M^R b_M^R)$ , where  $\Delta O a_M^R b_M^R$  is the reflection of  $\Delta O a_M b_M$  with regard to the  $y$ -axis. This leads to a unique decomposition:

$$(\tilde{w}; w_1, \dots, w_{\ell(\tilde{w})}), \quad \tilde{w} \in W_{N-M} \cup V_{N-M}, \quad w_i \in W_M, \quad i = 1, \dots, \ell(\tilde{w}) \quad (3.1)$$

such that  $\tilde{w}$  is similar to  $Q_M w$  and that the path segment  $(w(T_{i-1}^M(w)), w(T_{i-1}^M(w) + 1), \dots, w(T_i^M(w)))$  of  $w$  is identified with  $w_i \in W_M$  by appropriate rotation, translation and reflection so that  $w(T_{i-1}^M(w))$  is identified with  $O$  and  $w(T_i^M(w))$  with  $a_M$ . We shall use this kind of identification throughout the paper. We illustrate a simple example of the decomposition for  $N = 2$  and  $M = 1$  in Fig. 4.

### Erasure of the largest loops

- (1) Decompose a path  $w \in W_N \cup V_N$  into  $(\tilde{w}; w_1, \dots, w_{\ell(\tilde{w})})$ ,  $\tilde{w} = 2^{-(N-1)} Q_{N-1} w \in W_1 \cup V_1$ ,  $w_i \in W_{N-1}$   $i = 1, \dots, \ell(\tilde{w})$  as in (3.1) with  $M = N - 1$ . Fig. 5(a) shows the original  $w$  and Fig. 5(b) shows  $Q_{N-1} w$ .
- (2) Erase all the loops from  $\tilde{w}$  following the loop-erasure for  $W_1 \cup V_1$  to obtain  $L\tilde{w} \in \Gamma_1$ . Denote the coarse, loopless path  $2^{(N-1)} L\tilde{w}$  on  $F_{N-1}$  by  $\hat{Q}_{N-1} w$  (Fig. 5(c)).
- (3) Restore the original fine structures to the remaining parts as shown in Fig. 5(d) to obtain a path  $w' \in W_N \cup V_N$ . To be more precise, if we write  $\hat{Q}_{N-1} w = (w(T_0^{N-1}), w(T_{s_1}^{N-1}), \dots, w(T_{s_n}^{N-1}))$ , then for each  $i$ , between  $w(T_{s_i}^{N-1})$  and  $w(T_{s_{i+1}}^{N-1})$ , insert the path segment  $w_{s_i+1} = (w(T_{s_i}^{N-1}), w(T_{s_i}^{N-1} + 1), \dots, w(T_{s_{i+1}}^{N-1}))$  chosen from the original decomposition in Step (1). Note that  $Q_{N-1} w' = \hat{Q}_{N-1} w$  holds.

In this stage all the loops with diameter greater than  $2^{N-2}$  have been erased. We repeat Procedure (1)–(3) within each  $2^{N-1}$ -triangle to erase all the loops with diameter greater than  $2^{N-3}$ , and then within each  $2^{N-2}$ -triangle, and so on, until there remain no loops.

To describe the procedure more precisely, we prepare another kind of decomposition, a ‘triangle-based’ decomposition. For  $w \in W_N$  and  $0 \leq M \leq N$ , we shall define the sequence  $(\Delta_1, \dots, \Delta_k)$

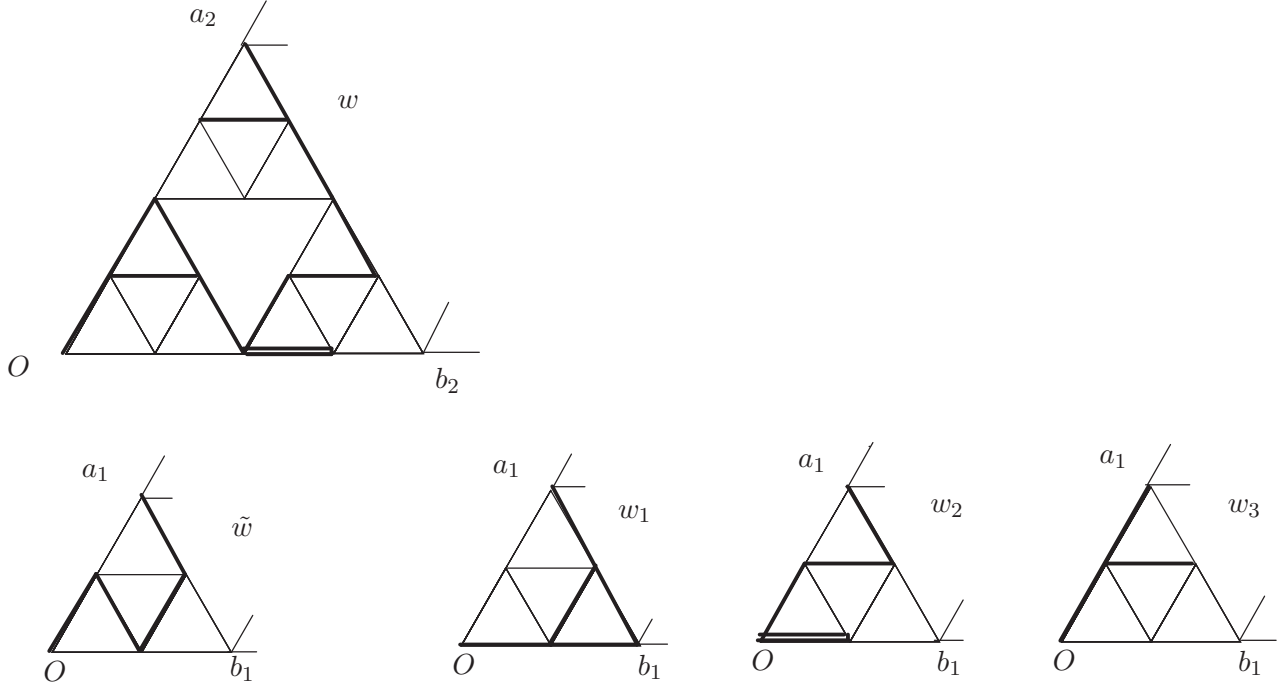


Fig 4:  $w, \tilde{w}, w_1, w_2, w_3$ .

of the  $2^M$ -triangles  $w$  ‘passes through’, and their exit times  $\{T_i^{ex,M}(w)\}_{i=1}^k$  as a subsequence of  $\{T_i^M(w)\}_{i=1}^m$  as follows: Let  $T_0^{ex,M}(w) = 0$ . There is a unique  $2^M$ -triangle that contains  $w(T_0^M)$  and  $w(T_1^M)$ , which we denote by  $\Delta_1$ . For  $i \geq 1$ , define

$$J(i) = \min\{j \geq 0 : j < m, T_j^M(w) > T_{i-1}^{ex,M}(w), w(T_{j+1}^M(w)) \notin \Delta_i\},$$

if the minimum exists, otherwise  $J(i) = m$ . Then define  $T_i^{ex,M} = T_i^{ex,M}(w) = T_{J(i)}^M(w)$ , and let  $\Delta_{i+1}$  be the unique  $2^M$ -triangle that contains both  $w(T_i^{ex,M})$  and  $w(T_{J(i)+1}^M)$ . By definition, we see that  $\Delta_i \cap \Delta_{i+1}$  is a one-point set  $\{w(T_i^{ex,M})\}$ , for  $i = 1, \dots, k-1$ . We denote the sequence of these triangles by  $\sigma_M(w) = (\Delta_1, \dots, \Delta_k)$ , and call it the  **$2^M$ -skeleton** of  $w$ . We call the sequence  $\{T_i^{ex,M}(w)\}_{i=0}^k$  **exit times** from the triangles in the skeleton. For each  $i$ , there is an  $n = n(i)$  such that  $T_{i-1}^{ex,M}(w) = T_n^M(w)$ . If  $T_i^{ex,M}(w) = T_{n+1}^M(w)$ , we say that  $\Delta_i \in \sigma_M(w)$  is **Type 1**, and if  $T_i^{ex,M}(w) = T_{n+2}^M(w)$ , **Type 2**. For  $w \in W_N \cup V_N$  and  $M < N$ , if  $Q_M w$  is similar to a path in  $\Gamma_{N-M}$ , namely,  $2^{-M}Q_M w \in \Gamma_{N-M}$ , then its  $2^M$ -skeleton is a collection of distinct  $2^M$ -triangles and each of them is either Type 1 or Type 2.

Assume  $w \in W_N \cup V_N$  and  $M \leq N$ . For each  $\Delta$  in  $\sigma_M(w)$ , the **path segment of  $w$  in  $\Delta$**  is defined by

$$w|_{\Delta} = [w(n), T_{i-1}^{ex,M}(w) \leq n \leq T_i^{ex,M}(w)]. \quad (3.2)$$

Note that the definition of  $T_i^{ex,M}(w)$  allows a path segment  $w|_{\Delta}$  to leak into the neighboring  $2^M$ -triangles. If  $Q_M w$  is similar to a path in  $\Gamma_{N-M}$ , then  $w|_{\Delta} \in W_M$  or  $w|_{\Delta} \in V_M$  (identification implied), according to the type of  $\Delta \in \sigma_M(w)$ , where the entrance to  $\Delta$  is identified with  $O$  and the exit with  $a_M$ . This means that each  $w$  such that  $Q_M w$  is similar to a path in  $\Gamma_{N-M}$  can be decomposed uniquely to

$$(\sigma_M(w); w|_{\Delta_1}, \dots, w|_{\Delta_k}), \quad w|_{\Delta_i} \in W_M \cup V_M, \quad i = 1, \dots, k. \quad (3.3)$$

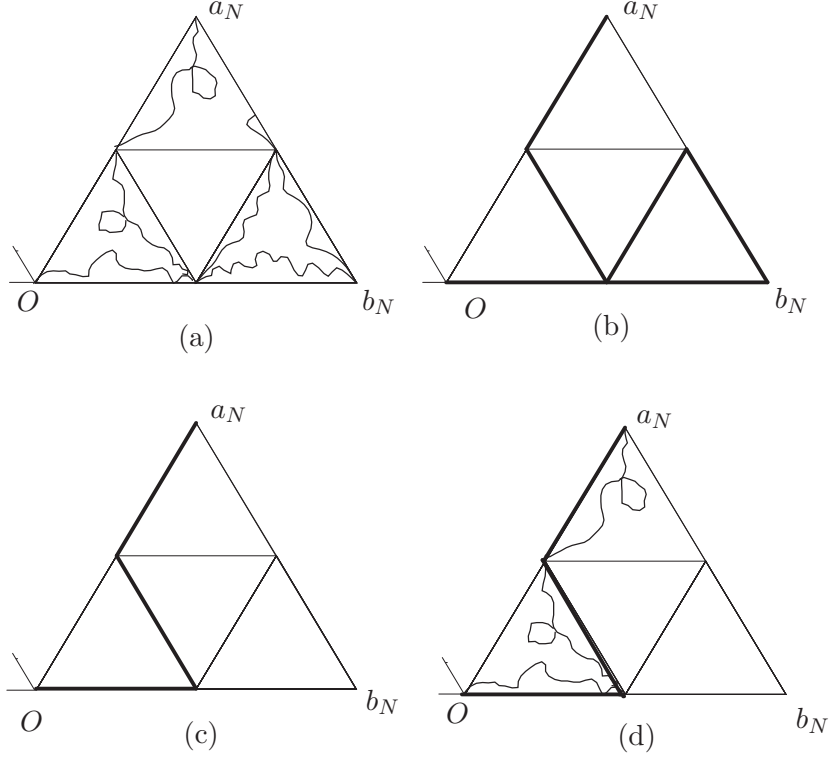


Fig 5: The loop-erasing procedure: (a)  $w$ , (b)  $Q_{N-1}w$ , (c)  $\hat{Q}_{N-1}w$ , (d) fine structures restored.

We call a loop  $[w(i), w(i+1), \dots, w(i+i_0)]$  a  **$2^M$ -scale loop** whenever there exists an  $M \in \mathbb{Z}_+$  such that

$$\max\{N' : w(i) = w(i+i_0) \in G_{N'}\} = M, \quad d \geq 2^M,$$

where  $d$  is the diameter of the loop.

### Induction step of loop erasure

Let  $w \in W_N \cup V_N$  and  $1 \leq M \leq N$ . Assume that all of the  $2^{N-1}$  to  $2^{N-M}$ -scale loops have been erased from  $w$ , and denote the path obtained at this stage by  $w' \in W_N \cup V_N$ . Note that  $Q_{N-M}w'$  is similar to a path in  $\Gamma_M$ .

- 1) Decompose  $w'$  to obtain  $(\sigma_{N-M}(w'); w'_1, \dots, w'_k)$ ,  $w'_i \in W_{N-M} \cup V_{N-M}$  as given in (3.3).
- 2) From each  $w'_i$ , erase  $2^{N-M-1}$ -scale loops (largest-scale loops) according to the base step procedure (1)–(3) above to obtain  $\tilde{w}'_i \in W_{N-M} \cup V_{N-M}$ .
- 3) Assemble  $(\sigma_{N-M}(w'); \tilde{w}'_1, \dots, \tilde{w}'_k)$  to obtain  $w'' \in W_N \cup V_N$ , which is determined uniquely.  $w''$  has no  $2^{N-1}$  to  $2^{N-M-1}$ -scale loops.

□

We repeat 1)–3) until we have no loops and denote the resulting loopless path  $Lw \in \Gamma_N$ . In this way, the loop erasing operator  $L$ , first defined for  $W_1 \cup V_1$ , has been extended to  $L : \bigcup_{N=1}^{\infty} (W_N \cup V_N) \rightarrow \bigcup_{N=1}^{\infty} \Gamma_N$  with  $L(W_N \cup V_N) = \Gamma_N$ . Note that the operation described above is essentially a repetition of loop-erasing for  $W_1 \cup V_1$ .

We induce measures  $\hat{P}_N = P_N \circ L^{-1}$  and  $\hat{P}'_N = P'_N \circ L^{-1}$ , which satisfy  $\hat{P}_N[\Gamma_N] = \hat{P}'_N[\Gamma_N] = 1$ . For  $w_1^*, \dots, w_{10}^*$  shown in Fig. 3, denote

$$p_i = \hat{P}_1[w_i^*] = P_1[w : Lw = w_i^*], \quad q_i = \hat{P}'_1[w_i^*] = P'_1[w : Lw = w_i^*].$$



They were obtained in [4] by direct calculation:

$$p_1 = 1/2, \quad p_2 = p_3 = p_7 = 2/15, \quad p_4 = p_5 = p_6 = 1/30, \quad p_8 = p_9 = p_{10} = 0, \quad (3.4)$$

$$q_1 = 1/9, \quad q_2 = q_3 = 11/90, \quad q_4 = q_5 = q_6 = 2/45, \quad q_7 = 8/45, \quad q_8 = 2/9, \quad q_9 = q_{10} = 1/18. \quad (3.5)$$

$\hat{P}_N$  and  $\hat{P}'_N$  define two kinds of walks  $Y_N = LZ_N$  and  $Y'_N = LZ'_N$  on  $F_0 \cap \triangle Oa_N b_N$  obtained by erasing loops from the simple random walks  $Z_N$  and  $Z'_N$ , respectively. We remark that  $\frac{2}{3}\hat{P}_N + \frac{1}{3}\hat{P}'_N$  equals to the ‘standard’ LERW studied in [14].

For  $w \in W_N \cup V_N$ , we defined  $\hat{Q}_{N-1}w$  in Step (2) for the erasure of the largest-scale loops. For later use we define  $\hat{Q}_{N-K}w$  on  $F_{N-K}$  for all  $K = 0, 1, \dots, N$ . Repeat the induction step 1)–3)  $K$  times to have down to  $2^{N-K}$ -scale loops erased and denote the resulting path  $w'$ . Let  $\hat{Q}_{N-K}w = Q_{N-K}w'$ , namely the coarse path before restoring fine structures. In particular,  $\hat{Q}_N w = Q_N w$  and  $\hat{Q}_0 w = Lw$ . By construction, the distributions of  $2^{-(N-K)}\hat{Q}_{N-K}Z_N$  and  $2^{-(N-K)}\hat{Q}_{N-K}Z'_N$  equal to  $\hat{P}_K$  and  $\hat{P}'_K$ , respectively.

## 4 Asymptotic behavior of the exit times

In this section, we look into the asymptotics of exit times  $T_1^{ex,N}(Y_N)$  and  $T_1^{ex,N}(Y'_N)$  as  $N \rightarrow \infty$ , which will be used in Section 6.

For  $w \in \Gamma_N$ , let us denote the number of  $2^0$ -triangles of Type 1 (the path passes two of the vertices) and those of Type 2 (the path passes all three vertices) in  $\sigma_0(w)$  by  $s_1(w)$  and  $s_2(w)$ , respectively. Note that  $T_1^{ex,N}(w) = \ell(w) = s_1(w) + 2s_2(w)$ . Define two sequences,  $\{\Phi_N^{(1)}\}_{N \in \mathbb{N}}$  and  $\{\Phi_N^{(2)}\}_{N \in \mathbb{N}}$ , of generating functions by:

$$\Phi_N^{(1)}(x, y) = \sum_{w \in \Gamma_N} \hat{P}_N(w) x^{s_1(w)} y^{s_2(w)},$$

$$\Phi_N^{(2)}(x, y) = \sum_{w \in \Gamma_N} \hat{P}'_N(w) x^{s_1(w)} y^{s_2(w)}, \quad x, y \geq 0.$$

For simplicity, we shall denote  $\Phi_1^{(1)}(x, y)$  and  $\Phi_1^{(2)}(x, y)$  by  $\Phi^{(1)}(x, y)$  and  $\Phi^{(2)}(x, y)$ . A crucial observation is that in the process of erasing loops from  $Z_{N+1}$ , if we stop at the point where we have obtained  $\hat{Q}_1 Z_{N+1}$  after erasing down to  $2^1$ -scale loops, it is nothing but the procedure for obtaining  $LZ_N$  from  $Z_N$ , namely, the distribution of  $2^{-1}\hat{Q}_1 Z_{N+1}$  equals to  $\hat{P}_N$ . The same holds for  $Z'_{N+1}$  as well. This combined with (3.4) and (3.5) leads to the recursion relations for the generating functions given below:

**Proposition 5** (*Proposition 3 in [4]*)

*The above generating functions satisfy the following recursion relations for all  $N \in \mathbb{N}$  :*

$$\Phi^{(1)}(x, y) = \frac{1}{30}(15x^2 + 8xy + y^2 + 2x^2y + 4x^3),$$

$$\Phi^{(2)}(x, y) = \frac{1}{45}(5x^2 + 11xy + 2y^2 + 14x^2y + 8x^3 + 5xy^2);$$

$$\Phi_{N+1}^{(i)}(x, y) = \Phi_N^{(i)}(\Phi^{(1)}(x, y), \Phi^{(2)}(x, y)), \quad i = 1, 2.$$

Define the mean matrix by

$$\mathbf{M} = \begin{bmatrix} \frac{\partial}{\partial x} \Phi^{(1)}(1, 1) & \frac{\partial}{\partial y} \Phi^{(1)}(1, 1) \\ \frac{\partial}{\partial x} \Phi^{(2)}(1, 1) & \frac{\partial}{\partial y} \Phi^{(2)}(1, 1) \end{bmatrix} = \begin{bmatrix} \frac{9}{5} & \frac{2}{5} \\ \frac{26}{15} & \frac{13}{15} \end{bmatrix}. \quad (4.1)$$

It is a strictly positive matrix, and the larger eigenvalue is given by  $\lambda = (20 + \sqrt{205})/15 = 2.2878\dots$ . The following is a restatement of Proposition 9 in [4].

**Proposition 6** (1) Let  $G_N^{(1)}(t)$  and  $G_N^{(2)}(t)$  be the Laplace transforms of  $\lambda^{-N}T_1^{ex,N}(Y_N)$  and  $\lambda^{-N}T_1^{ex,N}(Y'_N)$ , respectively, that is,

$$G_N^{(1)}(t) = \hat{E}_N[\exp(-t\lambda^{-N}T_1^{ex,N}(w))],$$

$$G_N^{(2)}(t) = \hat{E}'_N[\exp(-t\lambda^{-N}T_1^{ex,N}(w))], \quad t \in \mathbb{C}$$

where  $\hat{E}_N$  and  $\hat{E}'_N$  are expectations with regard to  $\hat{P}_N$  and  $\hat{P}'_N$ , respectively. Then they are expressed in terms of the generating functions as

$$G_N^{(i)}(t) = \Phi_N^{(i)}(e^{-\lambda^{-N}t}, e^{-2\lambda^{-N}t}) \quad i = 1, 2. \quad (4.2)$$

(2)  $\lambda^{-N}T_1^{ex,N}(Y_N)$  and  $\lambda^{-N}T_1^{ex,N}(Y'_N)$  converge in law to some integrable random variables  $T_1^*$  and  $T_2^*$ , respectively, as  $N \rightarrow \infty$ .  $T_1^*$  and  $T_2^*$  have strictly positive probability density functions on  $(0, \infty)$ .

(3) Let  $g_i(t)$  be the Laplace transform of  $T_i^*$ . For each  $i$ ,  $G_N^{(i)}(t)$  converges to  $g_i(t)$  uniformly on any compact set in  $\mathbb{C}$  as  $N \rightarrow \infty$ .  $g_1(t)$  and  $g_2(t)$  are entire functions on  $\mathbb{C}$  and the unique solution to

$$g_1(\lambda t) = \Phi^{(1)}(g_1(t), g_2(t)), \quad g_2(\lambda t) = \Phi^{(2)}(g_1(t), g_2(t)), \quad g_1(0) = g_2(0) = 1.$$

To obtain the left tail behavior of the scaled exit times, the following Tauberian theorem has a most suitable form.

**Theorem 7** (Theorem 5.9 in [6])

Let  $\mu_N$ ,  $N \in \mathbb{N}$  be a family of probability measures on  $[0, \infty)$  and let  $G_N(s) = \int_0^\infty e^{-sx} \mu_N(dx)$ ,  $s > 0$  be their Laplace transforms. If there exist positive constants  $C_{4.1} - C_{4.4}$ ,  $s_0 > 0$ ,  $s_1 \in \mathbb{R}$  and  $0 < \nu < 1$  such that

$$C_{4.1} \exp(-C_{4.2}s^\nu) \leq G_N(s) \leq C_{4.3} \exp(-C_{4.4}s^\nu),$$

holds for all  $s > s_0$  and  $N > s_1 + \frac{\nu}{\log 2} \log s$ , then the following holds.

(1) There exist positive constants  $C_{4.5}$  and  $C_{4.6}$  such that for any positive sequence satisfying

$$\lim_{N \rightarrow \infty} 2^{N(1-\nu)/\nu} \alpha_N = \infty \text{ and } \lim_{N \rightarrow \infty} \alpha_N = 0, \text{ the following holds:}$$

$$\begin{aligned} -C_{4.5} &\leq \liminf_{N \rightarrow \infty} \alpha_N^{\nu/(1-\nu)} \log \mu_N([0, \alpha_N]) \\ &\leq \limsup_{N \rightarrow \infty} \alpha_N^{\nu/(1-\nu)} \log \mu_N([0, \alpha_N]) \leq -C_{4.6}. \end{aligned}$$

(2) There exist positive constants  $C_{4.7} - C_{4.9}$  such that for any  $\xi > 0$  and  $N \in \mathbb{N}$  satisfying  $(2^{\frac{1}{\nu}-1})^N \xi \geq C_{4.7}$ ,

$$\mu_N([0, \xi]) \leq C_{4.8} e^{-C_{4.9} \xi^{-\nu/(1-\nu)}}$$

holds.

(1) is a kind of restatement of a Tauberian theorem of exponential type given in [8] and [9], and (2) is the combination of Chebyshev's inequality and (1).

**Proposition 8** For  $t > 0$ ,  $G_N^{(1)}(t)$  and  $G_N^{(2)}(t)$  satisfy the condition for Theorem 7 with  $\nu = \log 2 / \log \lambda$ .

*Proof.* Using (4.2), we rewrite the recursion as

$$G_{N+1}^{(i)}(t) = \Phi^{(i)}(G_N^{(1)}(t/\lambda), G_N^{(2)}(t/\lambda)), \quad i = 1, 2. \quad (4.3)$$

From the explicit form of  $\Phi^{(i)}$  in Proposition 5, we have for  $0 < x, y < 1$ ,

$$q_1(x \wedge y)^2 \leq \Phi^{(i)}(x, y) \leq (x \vee y)^2, \quad i = 1, 2,$$

where  $q_1 = 1/9$ . Repeating this  $M$  times, we have

$$\{q_1(x \wedge y)\}^{2^M} \leq \Phi_M^{(i)}(x, y) \leq (x \vee y)^{2^M}, \quad i = 1, 2. \quad (4.4)$$

This combined with (4.3) gives

$$\{q_1(G_N^{(1)}(t/\lambda^M) \wedge G_N^{(2)}(t/\lambda^M))\}^{2^M} \leq G_{N+M}^{(i)}(t) \leq \{G_N^{(1)}(t/\lambda^M) \vee G_N^{(2)}(t/\lambda^M)\}^{2^M} \quad (4.5)$$

Fix  $t_0 > 0$  arbitrarily. Since  $\{G_N^{(1)}(t_0) \vee G_N^{(2)}(t_0)\}_{N=1}^\infty$  and  $\{(G_N^{(1)}(\lambda t_0) \wedge G_N^{(2)}(\lambda t_0))\}_{N=1}^\infty$  are positive convergent sequences by Proposition 6 (3), there exist constants  $c_1, c_2 \in (0, 1)$  such that

$$q_1(G_N^{(1)}(\lambda t_0) \wedge G_N^{(2)}(\lambda t_0)) > c_1, \quad G_N^{(1)}(t_0) \vee G_N^{(2)}(t_0) < c_2, \quad (4.6)$$

for all  $N \in \mathbb{N}$ . For any  $t > t_0$ , choose  $M \in \mathbb{Z}_+$  such that

$$\lambda^M \leq \frac{t}{t_0} < \lambda^{M+1}. \quad (4.7)$$

Then, the monotonicity of  $G_N^{(i)}$  combined with (4.5), (4.6) and (4.7) gives

$$c_1^{2^M} \leq G_{N+M}^{(i)}(t) \leq c_2^{2^M}, \quad i = 1, 2.$$

This further leads to

$$\exp(-C_{4.2} t^\nu) \leq G_N^{(i)}(t) \leq \exp(-C_{4.4} t^\nu), \quad i = 1, 2$$

for all  $t > t_0$  and  $N > \log_\lambda(t/t_0)$ , where we put  $C_{4.2} = -\frac{\log c_1}{t_0^\nu}$  and  $C_{4.4} = -\frac{\log c_2}{2t_0^\nu}$ . □

## 5 Extention to the infinite Sierpiński gasket

In this section, we show that the loop-erased random walks defined in Section 3 can be extended to a loop-erased random walk on the infinite Sierpiński gasket. For this purpose, we need walks from  $O$  to  $b_N$  as well as those from  $O$  to  $a_N$ . For each  $N \in \mathbb{Z}_+$ , let

$$W_N^b = \{w = (w(0), w(1), \dots, w(n)) \in W : w(T_1^n(w)) = b_N, n = T_1^N(w)\},$$

$$V_N^b = \{w = (w(0), w(1), \dots, w(n)) \in W : w(T_1^N(w)) = a_N, w(T_2^N(w)) = b_N, n = T_2^N(w)\}.$$

and probability measures  $P_N^{(2)}$  on  $W_N^b$  and  $P_N^{(4)}$  on  $V_N^b$  by

$$P_N^{(2)}[w] = \left(\frac{1}{4}\right)^{\ell(w)-1}, \quad w \in W_N^b,$$

$$P_N^{(4)}[w] = \left(\frac{1}{4}\right)^{\ell(w)-2}, \quad w \in V_N^b.$$

Let  $U_N = W_N \cup V_N \cup W_N^b \cup V_N^b$  and extend the loop-erasing operator  $L$  to  $\bigcup_{N=1}^\infty U_N$ . Denote  $P_N^{(1)} = P_N$ ,  $P_N^{(3)} = P'_N$  and  $\hat{P}_N^{(i)} = P_N^{(i)} \circ L^{-1}$ , for  $i = 1, 2, 3, 4$ . In the rest of the paper, we use the same notation  $\Gamma_N$  for loopless paths in  $U_N$ . Let

$$\Omega = \{\omega = (\omega_0, \omega_1, \omega_2, \dots) : \omega_0 \in \Gamma_0, \omega_N \in \Gamma_N, \omega_N|_{N-1} = \omega_{N-1}, N \in \mathbb{N}\},$$

where  $\omega_N|_{N-1}$  denotes the path  $\omega_N$  stopped at  $T_1^{ex, N-1}(\omega_N)$  and  $\mathcal{B}$  the  $\sigma$ -algebra on  $\Omega$  generated by cylinder sets. Define the projection onto the first  $N+1$  elements by

$$\pi_N \omega = (\omega_0, \omega_1, \dots, \omega_N).$$

and a probability measure  $\tilde{P}_N$  on  $\pi_N \Omega$  by

$$\tilde{P}_N[(\omega_1, \dots, \omega_N)] = \frac{11}{28}(\hat{P}_N^{(1)}[\omega_N] + \hat{P}_N^{(2)}[\omega_N]) + \frac{3}{28}(\hat{P}_N^{(3)}[\omega_N] + \hat{P}_N^{(4)}[\omega_N]). \quad (5.1)$$

**Proposition 9** *The sequence  $\{\tilde{P}_N\}$ ,  $N \in \mathbb{Z}_+$  defined in (5.1) satisfies:*

$$\tilde{P}_N[(\omega_0, \omega_1, \dots, \omega_N)] = \sum_{\omega'} \tilde{P}_{N+1}[(\omega_0, \omega_1, \dots, \omega_N, \omega')], \quad (5.2)$$

where the sum is taken over all possible  $\omega' \in \Gamma_{N+1}$  such that  $\omega'|_N = \omega_N$ .

*Proof.* Assume  $u \in U_{N+1}$ . Recall that in Step (2) of erasing the largest-scale loops, namely,  $2^N$ -scale loops, from  $u$ , we obtain  $\hat{Q}_N u$ , which satisfies  $2^{-N} \hat{Q}_N u \in \Gamma_1$  and whose law under  $\hat{P}_{N+1}^{(i)}$  is equal to  $\hat{P}_1^{(i)}$ . Let  $\Delta_0 = \Delta O a_0 b_0$  and denote the path segment of  $2^{-N} \hat{Q}_N u$  in  $\Delta_0$  by  $u_1 := (2^{-N} \hat{Q}_N u)|_{\Delta_0}$ . Then  $u_1 \in \Gamma_0 = \{(O, a_0), (O, b_0), (O, b_0, a_0), (O, a_0, b_0)\}$ . Denote  $v_1^* = (O, a_0)$ ,  $v_2^* = (O, b_0)$ ,  $v_3^* = (O, b_0, a_0)$ ,  $v_4^* = (O, a_0, b_0)$ , and  $\Delta = \Delta O a_N b_N$ . For  $\hat{w} \in \Gamma_N$ , we classify the event  $\{u \in U_{N+1} : Lu|_\Delta = \hat{w}\}$  by  $u_1$ . Note that under the condition that  $u_1 = v_j^*$ , the distribution of  $Lu|_\Delta$  is equal to  $\hat{P}_N^{(j)}$ . Thus, for  $i = 1, 3$ ,

$$\begin{aligned} \hat{P}_{N+1}^{(i)}[w \in \Gamma_{N+1} : w|_\Delta = \hat{w}] &= P_{N+1}^{(i)}[u \in U_{N+1} : Lu|_\Delta = \hat{w}] \\ &= \sum_{j=1}^4 P_{N+1}^{(i)}[Lu|_\Delta = \hat{w} \mid u_1 = v_j^*] P_{N+1}^{(i)}[u_1 = v_j^*] \\ &= \sum_{j=1}^4 \hat{P}_N^{(j)}[\hat{w}] \hat{P}_1^{(i)}[v \in \Gamma_1 : v|_{\Delta_0} = v_j^*] \\ &= \hat{P}_N^{(1)}[\hat{w}] \hat{P}_1^{(i)}[\{w_1^*, w_3^*\}] + \hat{P}_N^{(2)}[\hat{w}] \hat{P}_1^{(i)}[\{w_5^*, w_7^*, w_8^*, w_9^*\}] \\ &\quad + \hat{P}_N^{(3)}[\hat{w}] \hat{P}_1^{(i)}[\{w_2^*, w_4^*\}] + \hat{P}_N^{(4)}[\hat{w}] \hat{P}_1^{(i)}[\{w_6^*, w_{10}^*\}]. \end{aligned}$$

Thus, we have

$$\hat{P}_{N+1}^{(1)}[w \in \Gamma_{N+1} : w|_{\Delta} = \hat{w}] = \frac{19}{30}\hat{P}_N^{(1)}[\hat{w}] + \frac{1}{6}\hat{P}_N^{(2)}[\hat{w}] + \frac{1}{6}\hat{P}_N^{(3)}[\hat{w}] + \frac{1}{30}\hat{P}_N^{(4)}[\hat{w}],$$

$$\hat{P}_{N+1}^{(3)}[w \in \Gamma_{N+1} : w|_{\Delta} = \hat{w}] = \frac{7}{30}\hat{P}_N^{(1)}[\hat{w}] + \frac{1}{2}\hat{P}_N^{(2)}[\hat{w}] + \frac{1}{6}\hat{P}_N^{(3)}[\hat{w}] + \frac{1}{10}\hat{P}_N^{(4)}[\hat{w}],$$

For  $i = 2$ , let  $\hat{w}^R$  and  $v_i^{*R}$  be the paths obtained by reflection of  $\hat{w}$  and  $v_i^*$  with regard to the line  $y = x$ , respectively. Then we have

$$\begin{aligned} \hat{P}_{N+1}^{(2)}[w \in \Gamma_{N+1} : w|_{\Delta} = \hat{w}] &= P_{N+1}^{(2)}[u \in U_{N+1} : Lu|_{\Delta} = \hat{w}] \\ &= \sum_{j=1}^4 P_{N+1}^{(2)}[Lu|_{\Delta} = \hat{w} \mid u_1 = v_j^*] P_{N+1}^{(2)}[u_1 = v_j^*] \\ &= \sum_{j=1}^4 P_{N+1}^{(1)}[Lu|_{\Delta} = \hat{w}^R \mid u_1 = v_j^{*R}] P_{N+1}^{(1)}[u_1 = v_j^{*R}] \\ &= \hat{P}_N^{(2)}[\hat{w}^R] \hat{P}_1^{(1)}[v \in \Gamma_1 : v|_{\Delta_0} = v_2^*] + \hat{P}_N^{(1)}[\hat{w}^R] \hat{P}_1^{(1)}[v \in \Gamma_1 : v|_{\Delta_0} = v_1^*] \\ &\quad + \hat{P}_N^{(4)}[\hat{w}^R] \hat{P}_1^{(1)}[v \in \Gamma_1 : v|_{\Delta_0} = v_4^*] + \hat{P}_N^{(3)}[\hat{w}^R] \hat{P}_1^{(1)}[v \in \Gamma_1 : v|_{\Delta_0} = v_3^*] \\ &= \hat{P}_N^{(1)}[\hat{w}] \hat{P}_1^{(i)}[\{w_5^*, w_7^*, w_8^*, w_9^*\}] + \hat{P}_N^{(2)}[\hat{w}] \hat{P}_1^{(i)}[\{w_1^*, w_3^*\}] \\ &\quad + \hat{P}_N^{(3)}[\hat{w}] \hat{P}_1^{(i)}[\{w_6^*, w_{10}^*\}] + \hat{P}_N^{(4)}[\hat{w}] \hat{P}_1^{(i)}[\{w_2^*, w_4^*\}]. \end{aligned}$$

Thus,

$$\hat{P}_{N+1}^{(2)}[w \in \Gamma_{N+1} : w|_{\Delta} = \hat{w}] = \frac{1}{6}\hat{P}_N^{(1)}[\hat{w}] + \frac{19}{30}\hat{P}_N^{(2)}[\hat{w}] + \frac{1}{30}\hat{P}_N^{(3)}[\hat{w}] + \frac{1}{6}\hat{P}_N^{(4)}[\hat{w}],$$

Similarly, we have

$$\hat{P}_{N+1}^{(4)}[w \in \Gamma_{N+1} : w|_{\Delta} = \hat{w}] = \frac{1}{2}\hat{P}_N^{(1)}[\hat{w}] + \frac{7}{30}\hat{P}_N^{(2)}[\hat{w}] + \frac{1}{10}\hat{P}_N^{(3)}[\hat{w}] + \frac{1}{6}\hat{P}_N^{(4)}[\hat{w}].$$

Thus, we see that

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left( \frac{11}{28}, \frac{11}{28}, \frac{3}{28}, \frac{3}{28} \right)$$

is the unique choice that satisfies

$$\sum_{i=1}^4 \alpha_i \hat{P}_{N+1}^{(i)}[w|_{\Delta} = \hat{w}] = \sum_{i=1}^4 \alpha_i \hat{P}_N^{(i)}[\hat{w}]$$

for every  $\hat{w} \in \Gamma_N$ ,  $N \in \mathbb{N}$ . □

Proposition 9 provides a consistency condition for Kolmogorov's extension theorem, and we have the unique probability measure  $P$  on  $(\Omega, \mathcal{B})$ , such that

$$P \circ \pi_N^{-1} = \tilde{P}_N.$$

For any  $n \in \mathbb{N}$ , take an  $N$  satisfying  $n \leq 2^N$ , then the distribution of the first  $n$  steps of the path,  $\omega_N|_n$  is uniquely determined independently of  $N$ .  $(\Omega, \mathcal{B}, P)$  defines a loop-erased random walk  $X$  on  $F_0$  such that for each  $\omega = (\omega_1, \omega_2, \dots)$  and  $i \in \mathbb{Z}_+$ ,

$$X(\omega)(i) = \omega_N(i), \quad i \leq 2^N.$$

This completes the proof of Theorem 1.

**Remark**

For  $N \in \mathbb{N}$  and  $w \in U_N$ , let  $u_M = (2^{-M} \hat{Q}_M w)|_{\Delta_0}$ ,  $M = 0, 1, \dots, N$ , where  $\hat{Q}_M$  is defined at the end of Section 3 and  $\Delta_0 = \triangle Oa_0b_0$ . Note that  $u_M \in \Gamma_0$ . For any  $M \leq N-1$  and any  $x_k \in \Gamma_0$ ,  $k = M, M+1, \dots, N$ ,

$$P_N^{(i)}[u_M = x_M \mid u_k = x_k, k = M+1, M+2, \dots, N] = P_N^{(i)}[u_M = x_M \mid u_{M+1} = x_{M+1}]. \quad (5.3)$$

Thus,  $P_N^{(i)}$ ,  $i = 1, 2, 3, 4$ ,  $N \in \mathbb{N}$  define a family of backward Markov chains on the state space  $\Gamma_0 = \{v_1^*, v_2^*, v_3^*, v_4^*\}$  such that

$$P_N^{(i)}[u_N = v_i^*] = 1,$$

and for  $M \leq N-1$ ,

$$P_N^{(i)}[u_M = v_j^* \mid u_{M+1} = v_k^*] = P_{kj},$$

where  $P_{kj}$  denotes the  $(k, j)$ -element of the transition probability matrix

$$\mathbf{P} = \frac{1}{30} \begin{bmatrix} 19 & 5 & 5 & 1 \\ 5 & 19 & 1 & 5 \\ 7 & 15 & 5 & 3 \\ 15 & 7 & 3 & 5 \end{bmatrix}.$$

$\alpha = \frac{1}{28}(11, 11, 3, 3)$  is the unique invariant probability vector, that is, the unique solution to

$$\alpha = \alpha P.$$

Moreover, for any probability vector  $a$ , it holds that

$$\lim_{n \rightarrow \infty} aP^n = \alpha.$$

In terms of the loop-erased walk measures, the above fact can be expressed as

$$\hat{P}_{N+K}^{(i)}[w|_K \in A_K] = \sum_{j=1}^4 (P^N)_{ij} \hat{P}_K^{(j)}[A_K],$$

where for  $w \in \Gamma_{N+K}$ ,  $w|_K$  denotes the path  $w$  stopped at  $T_1^{ex,K}(w)$  and  $A_K \subset \Gamma_K$ . Thus, for any probability vector  $a$ , we have as  $N \rightarrow \infty$ ,

$$\sum_{i=1}^4 a_i \hat{P}_{N+K}^{(i)}[w|_K \in A_K] \rightarrow \sum_{i=1}^4 \alpha_i \hat{P}_K^{(i)}[A_K].$$

In particular,  $\frac{1}{6}(2, 2, 1, 1)$  represents the ‘standard’ LERW studied in [14].

## 6 Proof of the theorems

Let  $X$  be the loop-erased random walk defined in Section 5 and let

$$\tilde{\Phi}_N(x, y) = \frac{11}{14} \Phi_N^{(1)}(x, y) + \frac{3}{14} \Phi_N^{(2)}(x, y), \quad (6.1)$$

where  $\Phi_N^{(i)}(x, y)$ ,  $i = 1, 2$  are defined in Section 4. The laplace transform of  $\lambda^{-N} T_1^{ex,N}(X)$  is given by

$$\tilde{g}_N(t) := \tilde{\Phi}_N(e^{-t\lambda^{-N}}, e^{-2t\lambda^{-N}}). \quad (6.2)$$

Define for each  $n \in \mathbb{N}$ ,

$$D_n(X) = \min\{M \geq 0 : |X(i)| \leq 2^M, 0 \leq i \leq n\},$$

and let  $K = K(n)$  be the positive integer such that

$$\lambda^K \leq n < \lambda^{K+1} \quad (6.3)$$

holds.

**Proposition 10** (*short-path estimate*) *There exist positive constants  $C_{6.1}$  and  $C_{6.2}$  such that*

$$P[ D_n(X) < K(n) - M ] \leq C_{6.1} e^{-C_{6.2} \lambda^M}$$

*holds for any  $n, M \in \mathbb{N}$  satisfying  $K(n) > M$ .*

*Proof.* Take  $C_{6.2} > 0$  arbitrarily. Since Proposition 6 (3) implies that  $\{\tilde{g}_N(t)\}$  is a convergent sequence for any  $t \in \mathbb{C}$ , we can take  $C_{6.1} > 0$  such that  $\tilde{g}_N(-C_{6.2}) < C_{6.1}$  for all  $N \in \mathbb{N}$ . By Chebyshev's inequality, we have

$$\tilde{P}_N[ \lambda^{-N} T_1^{ex, N}(X) \geq \lambda^M ] \leq \tilde{g}_N(-C_{6.2}) e^{-C_{6.2} \lambda^M} < C_{6.1} e^{-C_{6.2} \lambda^M}.$$

This leads to

$$\begin{aligned} P[ D_n(X) < K(n) - M ] &\leq P[ T_1^{ex, K-M}(X) > n ] \\ &= \tilde{P}_{K-M}[ T_1^{ex, K-M}(w) > n ] \\ &\leq \tilde{P}_{K-M}[ \lambda^{-(K-M)} T_1^{ex, K-M}(w) > \lambda^M ] \\ &\leq C_{6.1} e^{-C_{6.2} \lambda^M}. \end{aligned}$$

□

**Proposition 11** (*long-path estimate*) *There exist  $C_{6.3}, C_{6.4} > 0$  and  $N_0 \in \mathbb{N}$  such that*

$$P[ D_n(X) > K(n) + M ] \leq C_{6.3} e^{-C_{6.4} 2^M}$$

*for any  $n$  satisfying  $K(n) \geq N_0$  and any  $M \in \mathbb{N}$ .*

*Proof.* First note that

$$\begin{aligned} P[ D_n(X) > K(n) + M ] &\leq P[ T_1^{ex, K+M}(X) < n ] \\ &= \tilde{P}_{K+M}[ T_1^{ex, K+M}(w) < n ] \\ &\leq \tilde{P}_{K+M}[ T_1^{ex, K+M}(w) \leq \lambda^{K+1} ]. \end{aligned}$$

Fix  $0 < \delta < 1$  arbitrarily, then

$$\begin{aligned} \tilde{P}_{K+M}[ T_1^{ex, K+M}(w) \leq \lambda^{K+1} ] &= \sum_{w \in \Gamma_{K+M}, \ell(w) \leq \lambda^{K+1}} \tilde{P}_{K+M}[w] \\ &\leq \delta^{-1} \sum_{w \in \Gamma_{K+M}, \ell(w) \leq \lambda^{K+1}} \tilde{P}_{K+M}[w] \delta^{\ell(w) \lambda^{-(K+1)}} \\ &\leq \delta^{-1} \tilde{\Phi}_{K+M}(\delta^{\lambda^{-(K+1)}}, \delta^{2\lambda^{-(K+1)}}). \end{aligned}$$

Let  $t' = -\lambda^{-1} \log \delta > 0$ . Since Proposition 6 (3) implies that  $\tilde{\Phi}_N(\delta^{\lambda^{-(N+1)}}, \delta^{2\lambda^{-(N+1)}}) = \tilde{g}_N(t')$  converges as  $N \rightarrow \infty$  to a limit strictly smaller than 1. we can choose  $0 < r < 1$  and  $N_0 \in \mathbb{N}$  such that

$$\Phi_N^{(i)}(\delta^{\lambda^{-(N+1)}}, \delta^{2\lambda^{-(N+1)}}) < r, \quad i = 1, 2 \quad (6.4)$$

for all  $N \geq N_0$ . Thus if  $K \geq N_0$ ,

$$\tilde{\Phi}_{K+M}(\delta^{\lambda^{-(K+1)}}, \delta^{2\lambda^{-(K+1)}}) < \tilde{\Phi}_M(r, r) \leq r^{2^M} = e^{-C_{6.4}2^M},$$

where we used (4.4) in the last inequality and set  $C_{6.4} = -\log r$ . Taking  $C_{6.3} = \delta^{-1}$  completes the proof.  $\square$

To obtain the displacement exponent, we shall use the following inequality that holds for any  $\mathbb{N}$ -valued random variable  $Y$  and  $s > 0$ :

$$s C_{6.5}(s) \sum_{k=1}^{\infty} k^{s-1} P[ Y \geq k ] \leq E[Y^s] \leq s \sum_{k=1}^{\infty} k^{s-1} P[ Y \geq k ] + C_{6.6}(s). \quad (6.5)$$

For  $0 < s < 1$ ,  $C_{6.5}(s) = 1$ ,  $C_{6.6}(s) = 1$ , for  $s > 1$ ,  $C_{6.5}(s) = \frac{1}{2^s}$ ,  $C_{6.6}(s) = 0$  and  $C_{6.5}(1) = 1$ ,  $C_{6.6}(1) = 0$ .

Let  $\nu = \log 2 / \log \lambda$ .

**Proposition 12** *For any  $s > 0$ , there exist a positive constant  $C_1(s)$  and  $n_1 \in \mathbb{N}$  such that*

$$E[ |X(n)|^s ] \geq C_1(s) n^{s\nu},$$

for all  $n > n_1$ .

*Proof.* Fix  $M_0 \in \mathbb{N}$  such that  $C_{6.1}e^{-C_{6.2}\lambda^{M_0}} < 1/2$ , where  $C_{6.1}$  and  $C_{6.2}$  are as in Proposition 10. Take an  $n$  large enough so that  $K(n) > M_0 + 2$ , where  $K(n)$  is as in (6.3). Then

$$P[ |X(n)| \leq 2^{K-M_0-2} ] \leq P[ D_n < K - M_0 ] < \frac{1}{2}. \quad (6.6)$$

We give a proof in the case for  $s > 1$ . We make use of (6.5) with  $P[ |X(n)| > n ] = 0$  in mind.

$$\begin{aligned} E[ |X(n)|^s ] &\geq \frac{s}{2^s} \sum_{m=0}^{\infty} \sum_{k=2^{m+1}}^{2^{m+1}} k^{s-1} P[ |X(n)| \geq k ] \\ &\geq \frac{s}{2^s} \sum_{m=0}^{\infty} \sum_{k=2^{m+1}}^{2^{m+1}} (2^m)^{s-1} P[ |X(n)| > 2^{m+1} ] \\ &\geq s \sum_{m=0}^{\infty} 2^{s(m-1)} P[ |X(n)| > 2^{m+1} ] \\ &\geq s 2^{-(M_0+4)s} 2^{Ks} P[ |X(n)| > 2^{K-M_0-2} ] \\ &= s 2^{-(M_0+4)s} 2^{Ks} (1 - P[ |X(n)| \leq 2^{K-M_0-2} ]) \\ &\geq s 2^{-(M_0+5)s-1} 2^{Ks} \geq C_1(s) n^{s\nu}, \end{aligned}$$

where we used (6.6) and set  $C_1(s) = s 2^{-(M_0+5)s-1}$ . The case for  $0 < s \leq 1$  can be proved similarly.  $\square$



**Proposition 13** For any  $s > 0$ , there exist a positive constant  $C_2(s)$  and  $n_2 \in \mathbb{N}$  such that

$$E[ |X(n)|^s ] \leq C_2(s) n^{s\nu}$$

for all  $n > n_2$ .

*Proof.* First note that

$$P[ |X(n)| \geq 2^m ] \leq P[ D_n(X) > m - 1 ]. \quad (6.7)$$

Assume  $K = K(n) \geq N_0$  as in Proposition 11. In the case of  $s > 1$ , making use of (6.5), we have

$$\begin{aligned} E[ |X(n)|^s ] &\leq s \sum_{m=0}^{\infty} 2^m \cdot 2^{(s-1)(m+1)} P[ |X(n)| \geq 2^m ] \\ &\leq s 2^{s-1} \left( \sum_{m=0}^{K+1} 2^{sm} P[ |X(n)| \geq 2^m ] + \sum_{m=K+2}^{\infty} 2^{sm} P[ |X(n)| \geq 2^m ] \right) \\ &\leq s 2^{s-1} \left( \sum_{m=0}^{K+1} 2^{sm} + \sum_{m=K+2}^{\infty} 2^{sm} P[ D_n(X) > m - 1 ] \right) \quad (\text{use of (6.7)}) \\ &\leq c_1(s) 2^{Ks} + s 2^{2s-1} C_{6.3} 2^{Ks} \sum_{\ell=1}^{\infty} 2^{\ell s} e^{-C_{6.4} 2^\ell} \quad (\text{Proposition 11}) \\ &\leq C_2(s) n^{s\nu}, \end{aligned}$$

where  $c_1(s)$  and  $C_2(s)$  are positive constants depending only on  $s$  and we used the convergence of the series above. The case for  $0 < s \leq 1$  can be proved similarly.  $\square$

Proposition 13 combined with Proposition 12 gives Theorem 2.

Now we go on to prove the law of the iterated logarithm. First we prove the upper bound:

**Proposition 14** There exists  $C_4 > 0$  such that

$$\overline{\lim}_{n \rightarrow \infty} \frac{|X(n)|}{\psi(n)} \leq C_4, \quad P\text{-a.s.},$$

where  $\psi(n) = n^\nu (\log \log n)^{1-\nu}$ .

*Proof.* Let  $\mu_N$  be the distribution of  $\lambda^{-N} T_1^{ex, N}(X)$  under  $P$ . For each  $x > 1$  there is a unique integer  $N$  such that  $2^N \leq x < 2^{N+1}$ . For  $k > 0$  satisfying  $2^{-N} k \geq C_{4.7}$ , Theorem 7 (2) implies that

$$\begin{aligned} P[ \max_{0 \leq j \leq k} |X(j)| > x ] &\leq P[ T_1^{ex, N}(X) \leq k ] \\ &= \mu_N([0, \lambda^{-N} k]) \\ &\leq C_{4.8} e^{-C_{4.9}(xk^{-\nu}/2)^{1/(1-\nu)}}. \end{aligned}$$

Let  $\gamma > 1$  be arbitrary. For  $A > 0$ , let  $x = A\psi(\gamma^m)$  and  $k$  be the largest integer that does not exceed  $\gamma^{m+1}$ . The condition  $2^{-N} k \geq C_{4.7}$  is satisfied for  $m$  large enough. Thus, the above inequality leads to

$$\begin{aligned} \sum_{m=1}^{\infty} P[ \max_{\gamma^m < j \leq \gamma^{m+1}} |X(j)| > A\psi(\gamma^m) ] &\leq \sum_{m=1}^{\infty} P[ \max_{0 \leq j \leq \gamma^{m+1}} |X(j)| > A\psi(\gamma^m) ] \\ &\leq c_3 + C_{4.8} \sum_{m=1}^{\infty} e^{-C_{4.9}(xk^{-\nu}/2)^{1/(1-\nu)}} \\ &\leq c_3 + c_4 C_{4.8} \sum_{m=1}^{\infty} \frac{1}{m^\alpha}, \end{aligned}$$

for some constants  $c_3, c_4 > 0$  and  $\alpha = C_{4.9} \left( \frac{A}{2\gamma^\nu} \right)^{1/(1-\nu)}$ . The sequence  $\sum_{m=1}^{\infty} \frac{1}{m^\alpha}$  converges if we take  $A$  large enough so that  $\alpha > 1$ . The rest is a usual Borel-Cantelli argument and the statement holds with  $C_4 = A$ .  $\square$

Now we show the lower bound:

**Proposition 15** *There exists  $C_3 > 0$  such that*

$$C_3 \leq \overline{\lim}_{n \rightarrow \infty} \frac{|X(n)|}{\psi(n)}, \quad P\text{- a.s.}$$

*holds.*

The proof goes along the line of the argument used in [3], but we need to show how the ELLF construction enables us to make use of a ‘Markov structure’ to obtain the result. We use the following lemma:

**Lemma 16** *(A version of the second Borel-Cantelli Lemma used in [2]) Let  $B_1, B_2, \dots$  be a sequence of events and assume*

$$P[B_m \mid B_{m+1}^c, B_{m+2}^c, \dots, B_{m+k}^c] = P[B_m \mid B_{m+1}^c],$$

*for all  $m, k \in \mathbb{N}$ . Then*

$$\sum_{m=1}^{\infty} P[B_m \mid B_{m+1}^c] = \infty$$

*implies*

$$P[\overline{\lim}_{m \rightarrow \infty} B_m] = 1.$$

*Proof of Proposition 15.*

Let  $\beta = (1 - \nu)/\nu$ ,  $0 < b < 1$  and

$$A_M := \{\lambda^{-M} T_1^{ex, M}(X) \leq (b \log M)^{-\beta}\}.$$

We want to show that for an appropriate choice of  $b$ ,

$$P[\overline{\lim}_{M \rightarrow \infty} A_M] = 1$$

holds. Let  $S_j^{M-1}(X) = T_j^{ex, M-1}(X) - T_{j-1}^{ex, M-1}(X)$ ,  $j \in \{1, 2, 3\}$ , then

$$T_1^{ex, M}(X) = \sum_{j=1}^{|\sigma_{M-1}|} S_j^{M-1}(X), \quad (6.8)$$

where  $|\sigma_{M-1}|$  denotes the number of  $2^{M-1}$ -triangles in the  $2^{M-1}$ -skeleton of  $X$  stopped at  $T_1^{ex, M}(X)$ , which is either 2 or 3. Since the right-hand side contains only  $S_j^{M-1}$ ,  $j \in \{1, 2, 3\}$ , we have

$$P[A_M \mid A_{M-1}^c, A_{M-2}^c, \dots, A_{M-k}^c] = P[A_M \mid A_{M-1}^c], \quad (6.9)$$

and by repeated use of the definition of conditional probability combined with (6.9), we have

$$P[A_M \mid A_{M+1}^c, A_{M+2}^c, \dots, A_{M+k}^c] = P[A_M \mid A_{M+1}^c].$$

In order to show  $\sum_{M=1}^{\infty} P[A_M \mid A_{M+1}^c] = \infty$ , it is sufficient to show  $\sum_{M=1}^{\infty} P[A_M \cap A_{M+1}^c] = \infty$ . Let  $x_M = (b \log M)^{-\beta}$ , then since  $S_1^M(X) = T_1^{ex,M}(X)$ ,

$$\begin{aligned} P[A_M \cap A_{M+1}^c] &= \tilde{P}_{M+1}[T_1^{ex,M}(w) \leq \lambda^M x_M, T_1^{ex,M+1}(w) > \lambda^{M+1} x_{M+1}] \\ &= \sum_{y \leq \lambda^M x_M} \tilde{P}_{M+1}[T_1^{ex,M}(w) = y, \sum_{i \geq 2} S_i^M(w) + y > \lambda^{M+1} x_{M+1}] \\ &\geq \sum_{y \leq \lambda^M x_M} \tilde{P}_{M+1}[T_1^{ex,M}(w) = y, S_2^M + y > \lambda^{M+1} x_{M+1}]. \end{aligned}$$

Let  $\tilde{Z}_N$  be the simple random walk  $\tilde{P}_N$  defines on  $F_0 \cap \Delta O a_N b_N$ . Recall the procedure for loop erasure: after erasing largest-scale loops from  $\tilde{Z}_{M+1}$  in Step (2), we get  $\hat{Q}_M \tilde{Z}_{M+1}$  and the law of  $2^{-M} \hat{Q}_M \tilde{Z}_{M+1}$  is equal to  $\tilde{P}_1$ . We restore the original fine structures to these remaining parts and continue loop erasure. For each  $\Delta_i$  in  $\sigma_M(\hat{Q}_M \tilde{Z}_{M+1})$ , if  $\Delta_i$  is Type 1 with regard to  $\hat{Q}_M \tilde{Z}_{M+1}$ , the rest of the procedure is the same as loop erasure for  $Z_M$  (modulo rotation and reflection), and if Type 2, the same as that for  $Z'_M$  ( $Z_M$  and  $Z'_M$  are defined in (2.1) and (2.2)). Conditioned on  $\hat{Q}_M \tilde{Z}_{M+1}$ , parts in different  $2^M$ -triangles are independent. Classifying by the types of  $\Delta_1$  and  $\Delta_2$  in  $\sigma_M(\hat{Q}_M \tilde{Z}_{M+1})$ , we have

$$\begin{aligned} &\tilde{P}_{M+1}[T_1^{ex,M}(w) = y, S_2^M(w) + y > \lambda^{M+1} x_{M+1}] \\ &= \hat{P}_M[T_1^{ex,M}(w) = y] \hat{P}_M[T_1^{ex,M}(w) + y > \lambda^{M+1} x_{M+1}] \tilde{P}_1[\{w_1^*, w_5^*, w_7^*\}] \\ &\quad + \hat{P}'_M[T_1^{ex,M}(w) = y] \hat{P}_M[T_1^{ex,M}(w) + y > \lambda^{M+1} x_{M+1}] \tilde{P}_1[\{w_2^*, w_6^*, w_{10}^*\}] \\ &\quad + \hat{P}_M[T_1^{ex,M}(w) = y] \hat{P}'_M[T_1^{ex,M}(w) + y > \lambda^{M+1} x_{M+1}] \tilde{P}_1[\{w_3^*, w_8^*, w_9^*\}] \\ &\quad + \hat{P}'_M[T_1^{ex,M}(w) = y] \hat{P}'_M[T_1^{ex,M}(w) + y > \lambda^{M+1} x_{M+1}] \tilde{P}_1[\{w_4^*\}]. \end{aligned}$$

Since  $x_{M+1}/x_M \rightarrow 1$  as  $M \rightarrow \infty$ , we can take  $0 < c < 1$  such that  $c\lambda^{M+1}x_{M+1} < \lambda^M x_M$  for all large enough  $M$ . Then we have

$$\begin{aligned} &\sum_{y \leq \lambda^M x_M} \hat{P}_M[T_1^{ex,M}(w) = y] \hat{P}_M[T_1^{ex,M}(w) + y > \lambda^{M+1} x_{M+1}] \\ &\geq \sum_{c\lambda^{M+1}x_{M+1} \leq y \leq \lambda^M x_M} \hat{P}_M[T_1^{ex,M}(w) = y] \hat{P}_M[T_1^{ex,M}(w) + y > \lambda^{M+1} x_{M+1}] \\ &\geq \hat{P}_M[T_1^{ex,M}(w) \in [c\lambda^{M+1}x_{M+1}, \lambda^M x_M]] \hat{P}_M[T_1^{ex,M}(w) > \lambda^{M+1}(1-c)x_{M+1}]. \end{aligned}$$

By Proposition 6 (2),  $\lambda^{-M}T_1^{ex,M}$  under  $\hat{P}_M$  and under  $\hat{P}'_M$  converge in law to  $T_1^*$  and  $T_2^*$ , respectively, as  $M \rightarrow \infty$ , which combined with the fact that  $x_M \rightarrow 0$  as  $M \rightarrow \infty$  leads to

$$\begin{aligned} \hat{P}_M[T_1^{ex,M}(w) > \lambda^{M+1}(1-c)x_{M+1}] &= \hat{P}_M[\lambda^{-M}T_1^{ex,M}(w) > \lambda(1-c)x_{M+1}] \\ &\geq \hat{P}_M[\lambda^{-M}T_1^{ex,M}(w) > 1] > \frac{1}{2}P[T_1^* > 1], \end{aligned}$$

for  $M$  large enough.

With similar argument for the other terms, we have

$$\begin{aligned} &P[A_M \cap A_{M+1}^c] \\ &> a\hat{P}_M[T_1^{ex,M}(w) \in [c\lambda^{M+1}x_{M+1}, \lambda^M x_M]] \tilde{P}_1[\{w_1^*, w_3^*, w_5^*, w_7^*, w_8^*, w_9^*\}] \\ &\quad + a\hat{P}'_M[T_1^{ex,M}(w) \in [c\lambda^{M+1}x_{M+1}, \lambda^M x_M]] \tilde{P}_1[\{w_2^*, w_4^*, w_6^*, w_{10}^*\}] \\ &= a\tilde{P}_M[T_1^{ex,M}(w) \in [c\lambda^{M+1}x_{M+1}, \lambda^M x_M]], \end{aligned}$$

where  $a = \frac{1}{2}(P[T_1^* > 1] \wedge P[T_2^* > 1]) > 0$ . Moreover,

$$\begin{aligned} \tilde{P}_M[ T_1^{ex,M}(w) \in [ c\lambda^{M+1}x_{M+1}, \lambda^M x_M ] ] \\ = \tilde{P}_M[ \lambda^{-M}T_1^{ex,M}(w) \in [0, x_M] ] \left( 1 - \frac{\tilde{P}_M[ \lambda^{-M}T_1^{ex,M}(w) \in [0, \lambda c x_{M+1}] ]}{\tilde{P}_M[ \lambda^{-M}T_1^{ex,M}(w) \in [0, x_M] ]} \right). \end{aligned}$$

Since  $x_M \rightarrow 0$  and  $2^{M(1-\nu)/\nu}x_M \rightarrow \infty$  as  $M \rightarrow \infty$ , Theorem 7(1) with  $\alpha_N = x_N$  implies that for large enough  $M$

$$\tilde{P}_M[ \lambda^{-M}T_1^{ex,M}(w) \in [0, x_M] ] \geq \exp(-C_{4.5}x_M^{-1/\beta}) = \exp(-C_{4.5}b \log M) = \frac{1}{M^{C_{4.5}b}}.$$

We use Theorem 7 (1) again to have

$$1 - \frac{\tilde{P}_M[ \lambda^{-M}T_1^{ex,M}(w) \in [0, \lambda c x_{M+1}] ]}{\tilde{P}_M[ \lambda^{-M}T_1^{ex,M}(w) \in [0, x_M] ]} \rightarrow 1$$

as  $M \rightarrow \infty$ , thus this factor is greater than  $1/2$  for large enough  $M$ . Choose  $0 < b < 1$  so that  $C_{4.5}b < 1$ , then for some  $M_0 > 0$  and  $c_5 > 0$ , it holds that

$$\sum_{M=1}^{\infty} P[A_M \cap A_{M+1}^c] \geq c_5 + \frac{1}{2} \sum_{M \geq M_0} \frac{1}{M^{C_{4.5}b}} = +\infty.$$

Lemma 16 implies that for almost all  $\omega \in \Omega$ , there exists an increasing sequence  $\{M_k(\omega)\}$ ,  $k = 1, 2, \dots$  such that

$$\lambda^{-M_k}T_1^{ex,M_k}(X) \leq (b \log M_k)^{-\beta}. \quad (6.10)$$

It follows that for  $M_k \geq 3$

$$M_k \geq \frac{\log T_1^{ex,M_k}(X) + \beta \log b}{\log \lambda} + \frac{\beta \log \log M_k}{\log \lambda} \geq \frac{\log T_1^{ex,M_k}(X) + \beta \log b}{\log \lambda},$$

and for any small  $\varepsilon > 0$ , there exists a  $k_0 \in \mathbb{N}$  such that

$$\log M_k \geq (1 - \varepsilon) \log \log T_1^{ex,M_k}(X) \quad (6.11)$$

holds for all  $k \geq k_0$ . On the other hand, (6.10) implies

$$|X(T_1^{ex,M_k}(X))| = 2^{M_k} \geq (b \log M_k)^{1-\nu} M_k^\nu.$$

This combined with (6.11) leads to

$$\overline{\lim}_{n \rightarrow \infty} \frac{|X(n)|}{\psi(n)} \geq b^{1-\nu}(1 - \varepsilon)^{1-\nu}.$$

Since  $\varepsilon$  is arbitrary, we have proved the proposition with  $C_3 = b^{1-\nu}$ . □

Proposition 14 combined with Proposition 15 gives Theorem 3.

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